CAT 5

cohomology

Def 0. Let V be a vector space over a field F. Its <u>DUAL SPACE</u> is

$$\boxed{\tilde{V} = \mathrm{Hom}(V; \mathbb{F})}$$

i.e., the (vector!) space of field homomorphisms from V to $\mathbb{F}$: Eg $\mathbb{F} = \mathbb{R}$, $V = \mathbb{R}^n$.

When V is finite-dimensional, we get $\dim(\tilde{V}) = \dim(V)$, every basis $B = \{e_i\}$ for V induces a <u>dual</u> basis $\tilde{B} = \{\tilde{e}_i\}$ for $\tilde{V}$ via

$$\tilde{B} = \left\{ \tilde{e}_i : V \to \mathbb{F} \;\middle|\; \underbrace{\tilde{e}_i(e_j)}_{\langle \tilde{e}_i, e_j \rangle} = \begin{cases} 1_{\mathbb{F}}, & i = j \\ 0_{\mathbb{F}} & \text{otherwise} \end{cases} \right\}$$

Prop 1 If $f : U \to V$ is a linear map, then $\exists !$ linear map $\tilde{f} : \tilde{V} \to \tilde{U}$ satisfying

$$\langle \tilde{v}, f(u) \rangle = \langle \tilde{f}(\tilde{v}), u \rangle \quad \forall \begin{array}{l} u \in U \\ \tilde{v} \in \tilde{V} \end{array}$$

[In matrix-speak, this is just the <u>transpose</u>. With $\mathbb{F} = \mathbb{C}$ coefficients, one needs a conjugate-transpose.]

COHOMOLOGY

Def 2 Given a chain complex over $\mathbb{F}$, say:

$$\cdots \longrightarrow C_3 \xrightarrow{d_3} C_2 \xrightarrow{d_2} C_1 \xrightarrow{d_1} C_0 \longrightarrow 0$$

its DUAL COCHAIN COMPLEX is obtained by attaching everything with "$\sim$":

$$\boxed{\cdots \leftarrow C_3 \xleftarrow{\tilde{d}_3} \tilde{C}_2 \xleftarrow{\tilde{d}_2} C_1 \xleftarrow{\tilde{d}_1} \tilde{C}_0 \leftarrow 0}$$

So, $\underset{\text{[COCHAIN GROUP]}}{\tilde{C}_i = \mathrm{Hom}(C_i, \mathbb{F})}$ and $\underset{\text{[COBOUNDARY MAP]}}{\tilde{d}_i = d_i^{\text{Transpose}}}$

Note 3

a) It is typical to differentiate cochain notation from chain notation by using superscript indices, i.e.,

$$C^i := \tilde{C}_i = \mathrm{Hom}(C_i, \mathbb{F})$$

and similarly for coboundary maps, but with an index-shift:

$$d^i : C^i \to C^{i+1} \qquad (\text{instead of } d_{i+1}^T)$$

b) Every cochain complex obtained as above,

$$0 \to C^0 \xrightarrow{d^0} C^1 \xrightarrow{d^1} C^2 \to \cdots$$

again satisfies the chain complex property $d \circ d = 0$: note that

$$d^i \circ d^{i-1} = (d_{i+1}^T \circ d_i^T)$$
$$= (d_i \circ d_{i+1})^T \underset{\substack{\uparrow \\ \text{(chain cpk)}}}{=} 0^T = 0 \quad !$$

Def 4

The <u>COHOMOLOGY</u> of a cochain complex $(0 \to C^0 \xrightarrow{d^0} C^1 \xrightarrow{d^1} \cdots)$ is defined to be the "usual" quotient

$$\boxed{H^i(C^\bullet; \mathbb{F}) = \ker d^i / \mathrm{im}\, d^{i-1}}$$

[$\ker d^i =: Z^i =$ "i-cocycles" and $\mathrm{im}\, d^{i-1} =: B^i =$ "i-coboundaries"]

Q5

How does the homology of a chain complex relate to the cohomology of its dual?

$$H_i = \ker d_i / \mathrm{im}\, d_{i+1}$$

$$\cdots \to C_{i+1} \xrightarrow{d_{i+1}} C_i \xrightarrow{d_i} C_{i-1} \to \cdots$$

$$\Big\Downarrow \ \mathrm{Hom}(-; \mathbb{F})$$

$$\cdots \leftarrow C^{i+1} \xleftarrow{d^i} C^i \xleftarrow{d^{i-1}} C^{i-1} \leftarrow$$

$$H^i = \ker d^i / \mathrm{im}\, d_{i-1}$$

Ans. Well, they have the same dimensions...

Prop 6

Let $(\cdots \to C_3 \xrightarrow{d_3} C_2 \xrightarrow{d_2} C_1 \xrightarrow{d_1} C_0 \to 0)$ be a chain complex over a field $\mathbb{F}$ and let $(0 \to C^0 \xrightarrow{d^0} C^1 \xrightarrow{d^1} \cdots)$ be its dual. Then for every dimension $i \geq 0$,

$$\boxed{\dim H_i(C_\bullet; \mathbb{F}) = \dim H^i(C^\bullet; \mathbb{F})}$$

Pf

$$\dim H_i = \dim \ker d_i - \dim \operatorname{im} d_{i-1}$$
$$= [\dim C_i - \operatorname{rank}(d_i)] - [\operatorname{rank}(d_{i-1})]$$

Using $d^i = d_{i-1}^{\text{transpose}}$, both have same rank, so:

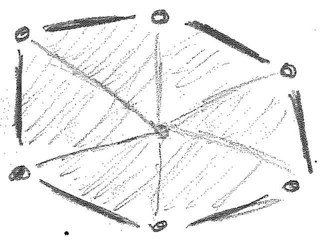
$$\dim H_i = [\dim C_i - \operatorname{rank}(d^{i+1})] - [\operatorname{rank}(d^i)]$$

Also, $C^i = \operatorname{Hom}(C_i, \mathbb{F})$ so $\dim C_i = \dim C^i$. So,

$$\dim H_i = [\dim C^i - \operatorname{rank}(d^i)] - [\operatorname{rank}(d^{i+1})]$$
$$= \dim(\ker d^i) - \operatorname{rank}(d^{i+1})$$
$$= \dim H^i$$

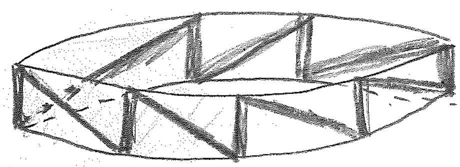
Fig 7

1-"cycle" =



→ each vertex in the boundary sum appears twice

1-"cocycle" =



→ each triangle in the coboundary sum appears twice

Def 8

(Assumes $C^\bullet$ comes from a simplicial complex)

There is a PRODUCT structure on cochains

$$C^i \times C^j \xrightarrow{\smile} C^{i+j}$$

defined as follows. (An i-cochain is a map $\varphi : C_i \to \mathbb{F}$, a j-cochain is a map $\psi : C_j \to \mathbb{F}$):

$$\boxed{\varphi \smile \psi([v_0, \ldots, v_{i+j}]) = \varphi([v_0, \ldots, v_i]) \cdot \psi([v_i, \ldots, v_j])}$$

$[v_0, \ldots, v_{i+j}]$: $(i+j)$-simplex, generates C_{i+j}

$\cdot$: product in $\mathbb{F}$.

(given $\tau = [v_0, \dots, v_{i+j}]$, let's write this as

$$\boxed{\varphi \cup \psi(\tau) = \varphi(\tau_{\le i}) \cdot \psi(\tau_{\ge i})}$$
)

Lem 9 Given φ and ψ as above, we have

$$\boxed{d^{i+j}(\varphi \cup \psi) = (d^i \varphi) \cup \psi + (-1)^j \cdot \varphi \cup (d^j \psi)}$$

Pf (Brute-force evaluation on $\tau = [v_0, \dots, v_{i+j}]$. Sorry!)

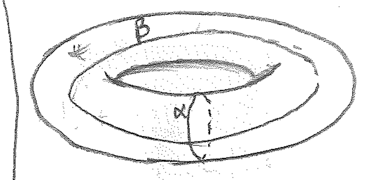
Def 10 The cochain product from Def 8 induces a well-defined product on cohomology, also denoted $\cup$:

(graded RING)

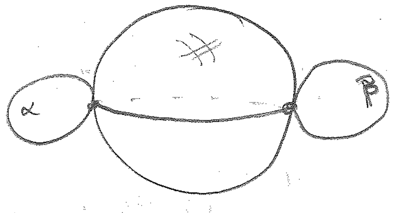
$$\boxed{H^i(C^\bullet; \mathbb{F}) \times H^j(C^\bullet; \mathbb{F}) \to H^{i+j}(C^\bullet; \mathbb{F})}$$

(meaning, $\cup$ sends cocycles/coboundaries to cocycles/coboundaries)
This is the CUP PRODUCT on cohomology.

Eg 11 The following spaces have the same homology/cohomology groups:



Torus



Sphere + Two circles (wedge product)

Both cases. $H_0 \simeq \mathbb{F} \simeq H_2$
$H_1 \simeq \mathbb{F}^2 = \langle \alpha, \beta \rangle$.

For the torus: $\alpha \cup \beta$ is the H^2-generator.
For the wedge product: $\alpha \cup \beta$ is zero.

So, the product structure on cohomology helps distinguish spaces that homology groups alone can NOT tell apart.

Note 12 A cochain map $f^\bullet : C^\bullet \to D^\bullet$ (can you define this?) induces well-defined morphisms on cohomology. And, the induced maps $H^i f : H^i(C^\bullet) \to H^i(D)$ are maps of rings! In other words, for each $\alpha \in H^i(C)$ and

β in $H^j(C^\circ)$, we have

$$\boxed{H^{i+j}f\,(\alpha \smile \beta) = H^i f(\alpha) \smile H^j f(\beta)}$$

(where $H^\circ f$ is the map induced by f on cohomology.

★ POINCARÉ DUALITY ★

There is a MUCH STRANGER product that "mixes" cohomology and homology. As before, let's fix a simplicial complex K and let $C_\circ$ & C° denote the usual chain and cochain complex of K.

Def 13 The CAP PRODUCT of a cochain $\varphi \in C^i$ with a chain in C_j is defined by the following action on each j-simplex τ:

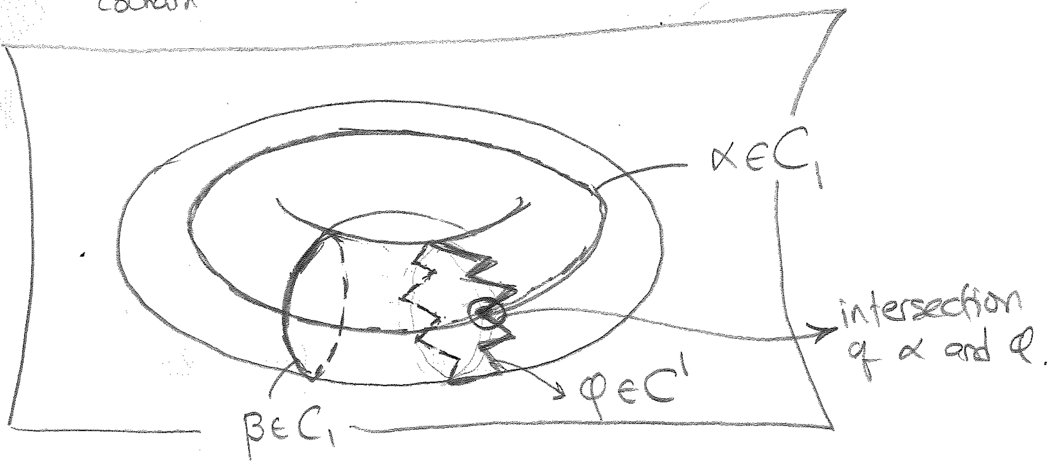
$$\varphi \frown \tau = \varphi(\tau_{\leq i}) \cdot \tau_{\geq i}$$

[Assumes $j \geq i$, otherwise zero]

[This produces a chain in C_{j-i}, so we have a map

$$\boxed{\underbrace{C^i}_{\text{cochain}} \times \underbrace{C_j}_{\text{chain}} \xrightarrow{\frown} \underbrace{C_{j-i}}_{\text{chain}}}$$

Eg 14



Here: $\varphi \frown \beta = 0$ but $\varphi \frown \alpha$ is a $\underset{\in C_0}{\underline{\text{vertex}}}$

[Cap products count intersections!]

Lem 15 The cap product satisfies

$$d_{j-i}(\varphi \frown \alpha) = (-1)^{i+1}\left[d^i\varphi \frown \alpha - \varphi \frown d_j \alpha\right]$$

for every $\varphi \in C^i$ and $\alpha \in C_j$

Pf Brute force, again.

Cor 16 The cap product sends

"cocycle" × "cycle" to "cycle"
"coboundary" × "cycle" to "boundary"

meaning that there are well-defined maps on (co)homology:

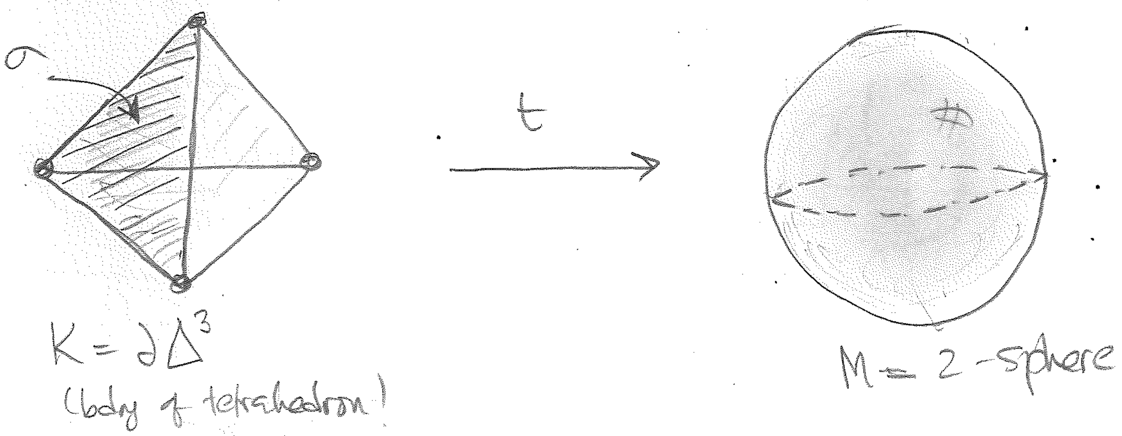
$$\boxed{H^i(K) \times H_j(K) \xrightarrow{\frown} H_{j-i}(K)}$$

Note 17

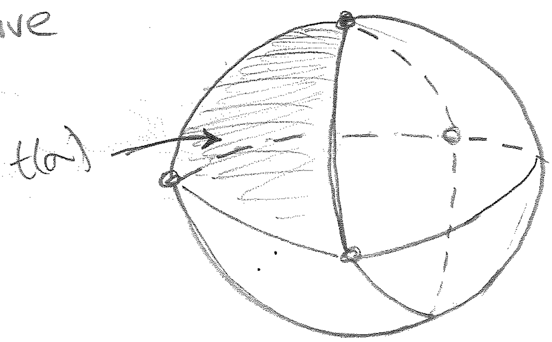
a) A <u>TRIANGULATION</u> of a manifold M is a homeomorphism

$$t : |K| \to M$$

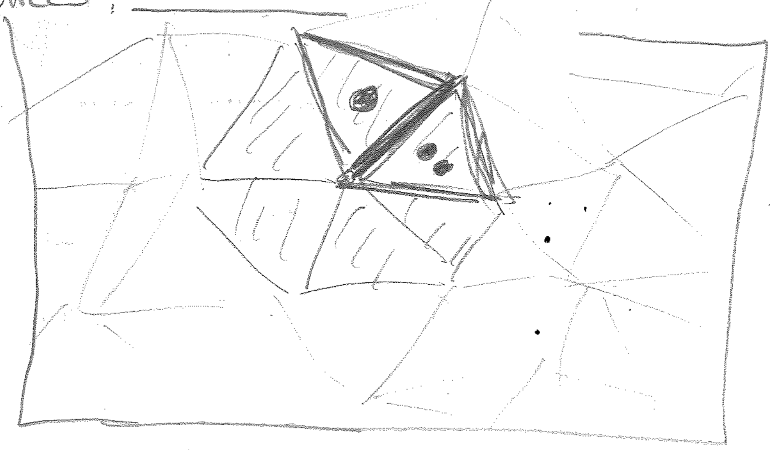
from the realization of a simplicial complex K to M. This decomposes M into pieces. eg.



might give



b). If M is n-dimensional, compact & triangulable, then one can find a triangulation so that every $(n-1)$ simplex lies in the boundary of EXACTLY two n-simplices:



c) We say M is ORIENTABLE if there exists a function

$$\theta : \left\{ \begin{matrix} n\text{-simplices} \\ \text{of } M \end{matrix} \right\} \to \{\pm 1_{\mathbb{F}}\}.$$

so that the chain

$$\boxed{[M] = \sum_{\substack{\sigma \text{ an } n\text{-splx} \\ \text{of } M}} \theta(\sigma)\cdot\sigma}$$

is a cycle.
(each $(n-1)$ simplex shows up twice in $\partial[M]$, with opposite sign)

This cycle $[M]$ is a generator (= basis element) of the top homology $H_n(M;\mathbb{F}) \cong \mathbb{F}$.

Thm 18 (Poincaré duality) Let M be a compact, n-dimensional, triangulated orientable manifold of dimension n. Then, the "cap product with $[M]$" map

$$\boxed{H^i(K) \xrightarrow{\frown[M]} H_{n-i}(K)}$$

is an isomorphism for each dimension $i \in \{0, \ldots, n\}$.

The proof is... complicated. But the consequences are wonderful. Here are two of them...

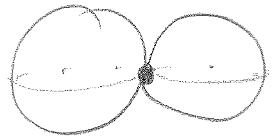
Cor 19 Every (triangulable, orientable compact n-dim) manifold M has palindromic Betti numbers:

$$\dim H_i(M) = \dim H^{m-i}(M) = \dim H_{m-i}(M).$$

(over a field $\mathbb{F}$)

Such as: • a torus: $(1,2,1)$

• a sphere $(1,0,\dots,0,1)$

But NOT  (a wedge of 2-spheres)

we get $(1,0,2)$.

Cor 20

a) The Euler characteristic of any odd-dimensional manifold is ZERO:

$$\chi(M) = \sum_{i=0}^{n} (-1)^i \dim H_i(M),$$

and the i-th term cancels the $(n-i)$-th one.

b) The Euler char of any $2k$-dimensional (≥ 0) manifold is even iff $\dim H_k$ is even.